

Endogenous Business Cycles in OLG Economies with Multiple Consumption Goods

Carine Nourry
Alain Venditti

WP 2014 - Nr 39

Endogenous business cycles in OLG economies with multiple consumption goods*

Carine NOURRY

*Aix-Marseille University (Aix-Marseille School of Economics), CNRS-GREQAM,
EHESS & Institut Universitaire de France*

and

Alain VENDITTI

*Aix-Marseille University (Aix-Marseille School of Economics), CNRS-GREQAM,
EHESS & EDHEC*

First version: *October 2010*; Revised: *March 2011*

Abstract: *We consider an OLG economy with two consumption goods. There are two sectors that produce a pure consumption good and a mixed good which can be either consumed or used as capital. We prove that the existence of Pareto optimal expectations-driven fluctuations is compatible with standard sectoral technologies if the share of the pure consumption good is low enough. Following Reichlin's (1986, *Journal of Economic Theory*, 40, 89-102) influential conclusion, this result suggests that some fiscal policy rules can prevent the existence of business-cycle fluctuations in the economy by driving it to the optimal steady state as soon as it is announced.*

Keywords: *Two-sector OLG model, multiple consumption goods, dynamic efficiency, endogenous fluctuations, local indeterminacy.*

Journal of Economic Literature Classification Numbers: C62, E32, O41.

*This paper is dedicated to Professor Kazuo Nishimura, a very close friend, who has made outstanding contributions to economic theory. This work was supported by French National Research Agency Grant (ANR-08-BLAN-0245-01). We thank H. d'Albis, R. Boucekine, J.M. Grandmont, T. Seegmuller, M. Yano and two anonymous referees for useful comments and suggestions that allowed us to greatly improve the presentation of our main results. Of course, any remaining errors or imprecisions are our own.

1 Introduction

A wide-spread perception among economists is that macroeconomic business-cycle fluctuations are not driven only by shocks on technologies or preferences, but also by changes in expectations about the fundamentals. A major strand of the literature focussing on fluctuations derived from agents' beliefs is based on the concept of sunspot equilibria that dates back to the early works of Shell [14], Azariadis [1] and Cass and Shell [3]. As shown by Woodford [16], the existence of sunspot equilibria is closely related to the equilibrium indeterminacy under perfect foresight, i.e. the existence of a continuum of equilibrium paths converging toward one steady state from the same initial value of the state variable.

Since the contribution of Reichlin [12], the possible co-existence of dynamic efficiency, *i.e.*, Pareto-optimal equilibrium paths, and local indeterminacy in OLG models has been widely discussed in the literature. This is an important question in terms of stabilization policies. Indeed, when local indeterminacy occurs under dynamic *efficiency*, the introduction of a public policy based on taxes and transfers could at the same time stabilize the economy and reach the Pareto optimal steady state along which all generations get an equal level of welfare. On the contrary, when local indeterminacy occurs under dynamic *inefficiency*, stabilization policies targeting the steady state leave room for welfare losses. Although Reichlin [12] has shown that locally indeterminate dynamically efficient equilibria can occur in an aggregate model with a Leontief technology, Cazzavillan and Pintus [4] have recently proved that this result is not robust to the introduction of any positive capital-labor substitution.

In Galor [8] type two-sector OLG models, with one pure consumption good and one pure investment good, the conclusion is not so clear-cut. Indeed, Drugeon *et al.* [7] and Nourry and Venditti [11] have proved that local indeterminacy is ruled out when the steady state is dynamically efficient provided the sectoral technologies are not too close to Leontief functions. The intuition of this result is quite simple. Starting from an arbitrary equilibrium, consider another one in which the agents expect a higher rate of investment at time t leading to some higher capital stock at time $t + 1$. This expectation will be self-fulfilling provided the amount of saving at date t is large enough to support the increase of the investment good output which directly provides the capital stock of the next period. When the equilibrium is dynamically efficient, the share of first period consumption is large enough to generate a stationary capital stock lower than the Golden Rule and thus prevents the agent from saving enough. At the same time, when the sectoral

technologies have elasticities of capital-labor substitution far enough from zero, any transfer of capital in the investment good sector is followed by a decrease of labor so that the final output cannot generate a large enough increase of capital.¹ As a result, the initial expectation cannot be realized as an equilibrium, and, under dynamic efficiency, local indeterminacy together with fluctuations based on local sunspots are ruled out.

However, in a multisector framework, the assumption of a unique consumption good is highly peculiar and is likely to generate singular properties. When heterogeneous sectors are introduced, it is quite common to consider instead that multiple consumption goods co-exist.² In such a case, the existence of additional substitution mechanisms between the different consumption goods suggests that new conclusions could be obtained. Unfortunately, the literature on OLG models has almost exclusively focused on Galor-type formulations. Among the few contributions dealing with a different framework, the most noteworthy contribution is the paper of Kalra [10] which studies the existence of cyclical equilibria in a generalization of the standard two-sector model by assuming that both goods are consumed.³ Beside the pure consumption good, the second sector produces a mixed good which can be either consumed or used as capital. Such a formulation, which is much in line with standard national accounting data than the usual Galor-type formulation,⁴ allows for the existence of intratemporal substitution in consumption. Kalra then shows that local indeterminacy through the occurrence of a Hopf bifurcation is more likely to occur than in the standard Galor-type model.⁵ However, as he does not discuss the dynamic efficiency property of the equilibrium, nothing is said about the question initially raised by Reichlin [12].

The purpose of this paper is then to complement the analysis of Kalra

¹On the contrary, when the technologies are close to Leontief, capital and labor are complement and both can be transferred in the investment good sector so that the increase of the final output may compensate the lack of savings and lead to an increase of capital compatible with the expectations. As shown in Nourry and Venditti [11], local indeterminacy then becomes compatible with dynamic efficiency.

²See for instance Benhabib and Nishimura [2] for an optimal growth infinite-horizon model with many consumption goods.

³See also Drugeon [6] for the analysis of saddle-point stability in a similar model with endogenous labor.

⁴National accounting data are based on subdivisions of the productive sectors in which many goods are both final and intermediary.

⁵For instance Reichlin [13] proves the existence of a Hopf bifurcation in a standard OLG model with Leontief technologies and Kalra [10] extends this conclusion to the case of technologies with substitutable factors.

and to explore the existence of dynamically efficient endogenous fluctuations. A simple intuition suggests indeed that new conclusions can be obtained. With two consumption goods, everything else equal, the stock of capital in the next period supported by the amount of saving at date t is lower than in the standard case since it is based on a mixed good output decreased by the part which is used for consumption. In such a case, a given amount of saving may be compatible with the expectation of a higher capital stock at time $t + 1$ if a larger part of the mixed good output is consumed. Based on this new channel, local indeterminacy and the existence of expectations-driven fluctuations can become compatible with dynamic efficiency under standard sectoral technologies with larger elasticities of capital-labor substitution.

We consider a simplified version of the Kalra's model assuming a unitary elasticity of intratemporal substitution between the two consumption goods. We provide a simple condition for dynamic efficiency. Then we show that, according to the previous intuition, when the pure consumption good is capital intensive, local indeterminacy and expectations-driven fluctuations occur for dynamically efficient competitive equilibria with larger sectoral elasticities of capital-labor substitution than in the standard model, provided the share of the pure consumption good in utility is low enough. We thus show that Reichlin's [12] result is more robust in a two-sector OLG economy with multiple consumption goods than in the standard Galor-type formulation.

The paper is organized as follows: Section 2 presents the model. Section 3 proves the existence of a steady state and provides a condition for dynamic efficiency. Section 4 contains our main results on the co-existence of local indeterminacy and dynamic efficiency. Some results under dynamic inefficiency are also provided in order to establish a better understanding of the relationship between the Pareto optimality and the dynamic properties of competitive equilibria. Concluding comments are in Section 5 and the proofs are gathered in a final Appendix.

2 The model

2.1 The production side

We consider an economy with two produced goods, one pure consumption good y_0 which cannot be used as capital, and one mixed good y which can be either consumed or invested, i.e. used as capital. There are two inputs, capital and labor. We assume complete depreciation of capital within one period and that labor is inelastically supplied. We then get

$$y_t - z_t = k_{t+1} \tag{1}$$

with z_t the consumption part of the mixed good in period t , and k_{t+1} the total amount of capital in period $t + 1$.

Each good is produced with a constant returns to scale technology such that $y_0 = f^0(k^0, l^0)$ and $y = f^1(k^1, l^1)$, with $k^0 + k^1 \leq k$, k being the total stock of capital, and $l^0 + l^1 \leq \ell$, ℓ being the total amount of labor.

Assumption 1. *Each production function $f^i : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, $i = 0, 1$, is C^2 , increasing in each argument, concave, homogeneous of degree one and such that for any $x > 0$, $f_1^i(0, x) = f_2^i(x, 0) = +\infty$, $f_1^i(+\infty, x) = f_2^i(x, +\infty) = 0$.*

For any given (k, y, ℓ) , profit maximization in the representative firm of each sector is equivalent to solving the following problem of optimal allocation of productive factors between the two sectors:

$$\begin{aligned} \tau(k, y, \ell) = \max_{k^0, k^1, l^0, l^1 \geq 0} \quad & f^0(k^0, l^0) \\ \text{s.t.} \quad & y \leq f^1(k^1, l^1), \quad k^0 + k^1 \leq k \text{ and } l^0 + l^1 \leq \ell \end{aligned} \quad (2)$$

The social production function $\tau(k, y, \ell)$ gives the maximal output of the consumption good. Under Assumption 1, $\tau(k, y, \ell)$ is homogeneous of degree one, concave and twice continuously differentiable.⁶ Denoting w the wage rate, r the gross rental rate of capital and p the price of the mixed good, all in terms of the price of the pure consumption good, we derive

$$r = \tau_1(k, y, \ell), \quad p = -\tau_2(k, y, \ell), \quad w = \tau_3(k, y, \ell) \quad (3)$$

2.2 The consumption side

In each period t , N_t agents are born, and they live for two periods. In their first period of life (when *young*), the agents are endowed with one unit of labor that they supply inelastically to firms. Their income is equal to the real wage. They allocate this income between current consumption and savings which are invested in the firms. In their second period of life (when *old*), they are retired and their income resulting from the return on the savings is entirely consumed. Each agent is assumed to have one child so that population is constant, *i.e.*, $N_t = N$.

The preferences of a representative agent born at time t are defined over his consumption bundle for each of the two produced goods, c_t^0 , c_t^1 , when young, and d_{t+1}^0 , d_{t+1}^1 , when old. They are summarized by the utility function $U(c_t^0, c_t^1, d_{t+1}^0, d_{t+1}^1) = u(C_t, D_{t+1}/B)$ with

$$C_t = (c_t^0)^\theta (c_t^1)^{1-\theta}, \quad D_{t+1} = (d_{t+1}^0)^\theta (d_{t+1}^1)^{1-\theta} \quad (4)$$

⁶See Benhabib and Nishimura [2].

and⁷

$$u(C_t, D_{t+1}/B) = \left[C_t^{1-1/\gamma} + \delta(D_{t+1}/B)^{1-1/\gamma} \right]^{\gamma/(\gamma-1)} \quad (5)$$

where $0 < \theta \leq 1$, $\delta > 0$ is the discount factor, $\gamma > 0$ is the elasticity of intertemporal substitution in consumption and $B > 0$ is a scaling parameter. C_t and D_{t+1} can be interpreted as composite goods derived from the two consumption goods y_t^0 and z_t with

$$y_t^0 = T(k_t, y_t, \ell_t) = N(c_t^0 + d_t^0) \text{ and } z_t = N(c_t^1 + d_t^1) \quad (6)$$

It follows that θ is the share of the pure consumption good and $1 - \theta$ the share of the mixed good in the composite goods C and D .

A young agent born in period t has first to solve two static problems of optimal composition of his two composite goods. Denoting by π_t the consumer price index in terms of the pure consumption good, we get the following optimization program for C_t

$$\max_{c_t^0, c_t^1 \geq 0} (c_t^0)^\theta (c_t^1)^{1-\theta} \text{ s.t. } c_t^0 + p_t c_t^1 = \pi_t C_t \quad (7)$$

The corresponding program for D_{t+1} is similar with $d_{t+1}^0 + p_{t+1} d_{t+1}^1 = \pi_{t+1} D_{t+1}$. Solving the associated first order conditions gives:

$$c_t^0 = \theta \pi_t C_t, \quad c_t^1 = \frac{(1-\theta)\pi_t C_t}{p_t}, \quad \pi_t = \left(\frac{p_t}{1-\theta} \right)^{1-\theta} \theta^{-\theta} \quad (8)$$

with similar expressions for d_{t+1}^0 and d_{t+1}^1 , namely

$$d_{t+1}^0 = \theta \pi_{t+1} D_{t+1}, \quad d_{t+1}^1 = \frac{(1-\theta)\pi_{t+1} D_{t+1}}{p_{t+1}} \quad (9)$$

Under perfect foresight, and considering w_t and R_{t+1} as given, a young agent also has to solve an intertemporal allocation problem in order to maximize his utility function over his life-cycle:

$$\max_{C_t, D_{t+1}, \phi_t \geq 0} u(C_t, D_{t+1}/B) \text{ s.t. } w_t = \pi_t C_t + \phi_t \text{ and } R_{t+1} \phi_t = \pi_{t+1} D_{t+1}$$

Solving the first order conditions gives:

$$C_t = \frac{w_t/\pi_t}{1 + \delta^\gamma [R_{t+1}\pi_t/(B\pi_{t+1})]^{\gamma-1}} \equiv \alpha(v_t) w_t/\pi_t \quad (10)$$

with $v_t = R_{t+1}\pi_t/(B\pi_{t+1})$ and $\alpha(v_t) \in (0, 1)$ the propensity to consume of the young, or equivalently the share of first period consumption spending over the wage income. We also get the saving function

$$\phi_t = \phi(w_t, v_t) \equiv (1 - \alpha(v_t)) w_t \quad (11)$$

In the rest of the paper we introduce the following standard Assumption:

Assumption 2. $\gamma > 1$

Such a restriction implies that the saving function (11) is increasing with respect to the gross rate of return R .

⁷All the results in this paper can be obtained with a general concave and homothetic utility function $u(C_t, D_{t+1}/B)$.

3 Dynamic efficiency of competitive equilibrium

3.1 Perfect-foresight competitive equilibrium

Total labor is given by the number of young households and is normalized to one, *i.e.*, $\ell = N = 1$. From now on, let $\tau(k, y, 1) = T(k, y)$ and $\tau_i(k, y, 1) = T_i(k, y)$, $i = 1, 2, 3$. Using (1), (3) and (11), a perfect-foresight competitive equilibrium is defined as a sequence $\{k_t, y_t\}_{t \geq 0}$ that satisfies $(1 - \alpha(R_{t+1}\pi_t/(B\pi_{t+1})))w_t = k_{t+1}$ and $y_t - z_t = k_{t+1}$ with $R_{t+1} = r_{t+1}/p_t$. Using (6), (8) and (9), we conclude that a perfect-foresight competitive equilibrium satisfies the following system of two difference equations:

$$\begin{aligned} k_{t+1} + \frac{T_3(k_t, y_t) \left[1 - \alpha \left(-\frac{T_1(k_{t+1}, y_{t+1})}{T_2(k_t, y_t)B} \left(\frac{T_2(k_t, y_t)}{T_2(k_{t+1}, y_{t+1})} \right)^{1-\theta} \right) \right]}{T_2(k_t, y_t)} &= 0 \\ k_{t+1} - \frac{y_t}{\theta} - \frac{(1-\theta)[T_3(k_t, y_t) + T_1(k_t, y_t)k_t]}{\theta T_2(k_t, y_t)} &= 0 \end{aligned} \quad (12)$$

with k_0 given.

It is worth noting at this point that if $\theta = 1$ the second difference equation reduces to $k_{t+1} = y_t$ and we get the standard two-sector OLG model with one pure consumption good and one pure investment good studied in Drugeon *et al.* [7] and Nourry and Venditti [11].⁸

3.2 A normalized steady state

A steady state $(k_t, y_t) = (k^*, y^*)$ for all t satisfies

$$\begin{aligned} k^* + \frac{T_3(k^*, y^*) \left[1 - \alpha \left(-\frac{T_1(k^*, y^*)}{T_2(k^*, y^*)B} \right) \right]}{T_2(k^*, y^*)} &= 0 \\ k^* - \frac{y^*}{\theta} - \frac{(1-\theta)[T_3(k^*, y^*) + T_1(k^*, y^*)k^*]}{\theta T_2(k^*, y^*)} &= 0 \end{aligned} \quad (13)$$

We consider a family of economies parameterized by the elasticity of intertemporal substitution in consumption γ . We follow the same procedure as in Drugeon *et al.* [7]: we use the scaling parameter B and the share θ to ensure the existence of a normalized steady state (NSS) (k^*, y^*) which remains invariant as γ is varied.

Let us define the maximum admissible value of capital \bar{k} solution of:

$$\bar{k} - f^1(\bar{k}, 1) = 0$$

Under Assumption 1 we have indeed $f^1(k, 1) > k$ if $k < \bar{k}$, while $f^1(k, 1) < k$ if $k > \bar{k}$. Obviously, the NSS must be such that $(k^*, y^*) \in (0, \bar{k}) \times (0, \bar{k})$. We then get:

⁸See also Galor [8] and Venditti [15].

Proposition 1. *Under Assumptions 1-2, let $(k^*, y^*) \in (0, \bar{k}) \times (0, \bar{k})$. Then there exist unique values $\theta(k^*, y^*) \in (0, 1)$ and $B(k^*, y^*, \gamma) > 0$ such that (k^*, y^*) is a steady state if and only if $\theta = \theta(k^*, y^*)$ and $B = B(k^*, y^*, \gamma)$.*

Proof: See Appendix 6.1. □

In the rest of the paper we assume $\theta = \theta(k^*, y^*)$ and $B = B(k^*, y^*, \gamma)$ so that the share of capital in total income, as given by

$$s = s(k^*, y^*) = \frac{r^* k^*}{T(k^*, y^*) + p^* y^*} \quad (14)$$

and $\alpha = \alpha(R^*/B(k^*, y^*, \gamma))$ remain constant as γ is made to vary.

3.3 Dynamic efficiency

From the homogeneity of τ , considering that $k^* T_2/T_3 = (T_2/T_1)(k^* T_1/T_3) = -s/(R(1-s))$, we derive the stationary gross rate of return along the NSS:

$$R^* = \frac{s}{(1-\alpha)(1-s)} \quad (15)$$

It is well-known that in OLG models if the capital-labor ratio exceeds the Golden-Rule level, the economy is dynamically inefficient. In our two-sector model, the Golden-Rule level, denoted \hat{k} , is characterized from the total stationary consumption which is given by the sum of the social production function with the consumption part of the mixed good, namely $C + D = T(k, y) + p[y - k]$. Denoting $R(k, y) = -T_1(k, y)/T_2(k, y)$, \hat{k} satisfies as usual $R(\hat{k}, \hat{y}) \equiv \hat{R} = 1$. Under-accumulation of capital is obtained if and only if $R^* > 1$. As in Drugeon *et al.* [7] we get:

Proposition 2. *Under Assumption 1, let $\underline{\alpha} = 1 - s/(1-s)$. Then an intertemporal competitive equilibrium converging towards the NSS is dynamically efficient if $\alpha \in (\underline{\alpha}, 1)$ and dynamically inefficient if $\alpha \in (0, \underline{\alpha})$.*

Dynamic inefficiency can be avoided if the amount of savings is not too large.

4 Endogenous business cycles

Our aim is to show that a two-sector OLG model with two consumption goods provides new results on the existence of endogenous business cycles derived from agents' beliefs. This type of fluctuations is based on the concept of sunspot equilibria. As shown by Woodford [16], the existence of sunspot equilibria is closely related to the equilibrium local indeterminacy, i.e. the existence of a continuum of equilibrium paths converging toward the NSS from the same initial capital stock. In our framework, local indeterminacy

occurs when the characteristic roots associated with the linearization of equations (12) around the NSS are less than 1 in absolute value.

We focus on two particular aspects of the model: the share θ of the pure consumption good in the composite goods C and D , and the share α of first period consumption spending in total income. Let us also introduce the relative capital intensity difference across sectors and the elasticity of the rental rate of capital, respectively

$$b \equiv \frac{l^1}{y} \left(\frac{k^1}{l^1} - \frac{k^0}{l^0} \right) \text{ and } \varepsilon_{rk} = -T_{11}(k^*, y^*)k^*/T_1(k^*, y^*) \quad (16)$$

Note that ε_{rk} is negatively linked to the sectoral elasticities of capital-labor substitution (see Drugeon [5]).

4.1 A labor intensive pure consumption good

Let us consider first the case $b > 0$. We derive from the homogeneity of τ and equations (3) that at the NSS $b < b_{Max}$ with

$$b_{Max} = \frac{s}{R^*[1-\theta[(1-s)\alpha+s]]} > 0 \quad (17)$$

We get the following results:

Proposition 3. *Under Assumptions 1-2, any equilibrium path is unique and monotone when the pure consumption good is labor intensive ($b \in (0, b_{Max})$).*

Proof: See Appendix 6.2. □

This Proposition implies that endogenous fluctuations and local indeterminacy are ruled out when $b \in (0, b_{Max})$. It complements the analysis of Kalra [10] in the case of a labor intensive pure consumption good. First it confirms that endogenous period-2 cycles through the existence of negative characteristic roots and a flip bifurcation are ruled out. Second, it proves that endogenous quasi-periodic cycles through the existence of a Hopf bifurcation are also ruled out in the case of a unitary elasticity of intratemporal substitution between the two consumption goods. Therefore, Proposition 2 in Kalra [10] requires an elasticity sufficiently lower than one.

4.2 A capital intensive pure consumption good

Let us consider from now on the case $b < 0$ and focus in a first step on dynamically efficient equilibria with $\alpha > \underline{\alpha}$. We assume $\theta < 1/2$ to provide conditions for the existence of local indeterminacy through the occurrence of a Hopf bifurcation. The motivation to consider such a configuration comes from the fact that when θ is close to 1, a Hopf bifurcation cannot occur

under dynamic efficiency (see Nourry and Venditti [11]). We also restrict the share of capital to get a positive value for the bound $\underline{\alpha} = 1 - s/(1 - s)$ and to be compatible with standard empirical values:

Assumption 3. $s \in [1/3, 1/2)$

Proposition 4. *Under Assumptions 1-3, let $\alpha > s/(1 - s)(> \underline{\alpha})$ and $b \in (-(1 - \alpha)s/[\alpha(1 - \theta + \theta s)], 0)$. Then there exist $\bar{\theta} \in (s/[(1 - s)\alpha + s], 1/2]$, $\tilde{\epsilon}_{rk} < \underline{\epsilon}_{rk} < \bar{\epsilon}_{rk}$ and $\gamma_H, \gamma_F > \gamma_T > 1$ such that when $\theta \in (s/[(1 - s)\alpha + s], \bar{\theta})$, the following results hold:*

1 - *If $\epsilon_{rk} \in (\underline{\epsilon}_{rk}, \bar{\epsilon}_{rk})$, then $\gamma_F > \gamma_H$ and the NSS is locally indeterminate when $\gamma \in (\gamma_T, \gamma_H)$ and undergoes a Hopf bifurcation when $\gamma = \gamma_H$. Moreover, there generically exist locally indeterminate (resp. locally unstable) quasi-periodic cycles when $\gamma \in (\gamma_H, \gamma_H + \epsilon)$ (resp. $\gamma \in (\gamma_H - \epsilon, \gamma_H)$) with $\epsilon > 0$, i.e. when the bifurcation is super (resp. sub-) critical.*

2 - *If $\epsilon_{rk} \in (\tilde{\epsilon}_{rk}, \underline{\epsilon}_{rk})$, then $\gamma_H > \gamma_F$ and the NSS is locally indeterminate when $\gamma \in (\gamma'_T, \gamma_F)$ and undergoes a flip bifurcation when $\gamma = \gamma_F$. Moreover there generically exist locally indeterminate (resp. saddle-point stable) period-two cycles when $\gamma \in (\gamma_F, \gamma_F + \varepsilon)$ (resp. $\gamma \in (\gamma_F - \varepsilon, \gamma_F)$) with $\varepsilon > 0$, i.e. when the bifurcation is super (resp. sub-) critical.*

Proof: See Appendix 6.3. □

Remark 1. *Whether the bifurcation is super or sub-critical is driven by the sign of some coefficient computed from the second and third order approximations of the dynamical system (12). This property determines whether the bifurcation leads to the occurrence of locally indeterminate or unstable (resp. saddle-point stable) quasi-periodic (resp. period-two) cycles near the bifurcation value.*

Proposition 4 shows that when the share θ of the pure consumption good in the composite goods is low enough, local indeterminacy and expectations-driven fluctuations arise when $\epsilon_{rk} \in (\tilde{\epsilon}_{rk}, \bar{\epsilon}_{rk})$, i.e. for strictly positive but intermediary values of the sectoral elasticities of capital-labor substitution. We then get more general conclusions for a wider range of elasticities of capital-labor substitution than in the case $\theta = 1$ in which the occurrence of local indeterminacy is only obtained through a flip bifurcation and requires to consider sectoral technologies very close to Leontief functions (see Nourry and Venditti [11]). The intuition for the existence of dynamically efficient expectations-driven fluctuations with multiple consumption goods given in the introduction is thus confirmed.

We thus show that Reichlin's [12] result is more robust in a two-sector OLG economy with multiple consumption goods than in the Galor-type formulation. The introduction of a public policy based on taxes and transfers can at the same time stabilize the economy and reach the Pareto optimal steady state along which all generations get an equal level of welfare.⁹

In order to establish a better understanding of the relationship between the Pareto optimality and the dynamic properties of competitive equilibria, let us finally focus on dynamically inefficient equilibria with $\alpha < \underline{\alpha}$. We successively consider the existence of local indeterminacy through flip and Hopf bifurcations. As mentioned in Remark 1, the two following Propositions will encompass the super and sub-critical bifurcations. Let us denote

$$\bar{b} = \min \left\{ \frac{1}{R^*(1-\theta)-\theta}, -\frac{(1-\alpha)s}{\alpha[1-\theta+\theta s]} \right\}$$

Proposition 5. *Under Assumptions 1-3, let $\alpha < \underline{\alpha}$. If one of the following conditions is satisfied:*

- i) $\theta \in (s/[1 - \alpha(1 - s)], 1/2)$ and $b < \bar{b}$,
 - ii) $\theta \in (1/2, [s^2 + \alpha(1 - s)]/[2s^2 + \alpha(1 - s)^2])$ and $b \in (1/[R^*(1 - 2\theta)], \bar{b})$,
- then there exists $\gamma_F > 1$ such that the NSS is locally indeterminate when $\gamma > \gamma_F$ and undergoes a flip bifurcation when $\gamma = \gamma_F$. Moreover, there generically exist locally indeterminate (resp. saddle-point stable) period-two cycles when $\gamma \in (\gamma_F - \varepsilon, \gamma_F)$ (resp. $\gamma \in (\gamma_F, \gamma_F + \varepsilon)$) with $\varepsilon > 0$, i.e. when the bifurcation is super (resp. sub-) critical.

Proof: See Appendix 6.4. □

When compared with Proposition 4, Proposition 5 shows that considering dynamically efficient equilibria implies restrictions for the existence of local indeterminacy that are not necessary under dynamic inefficiency. Indeed, expectations-driven fluctuations now occur without any condition on the elasticity of the rental rate of capital ε_{rk} and are thus compatible with any values of the sectoral elasticities of capital-labor substitution.

This conclusion is also confirmed when local indeterminacy is appraised through the existence of a Hopf bifurcation. Let us indeed denote

$$\tilde{\theta} = \frac{s+(1-\alpha)(1-s)+(1-s)\sqrt{(\underline{\alpha}-\alpha)(1-\alpha)}}{s+3(1-\alpha)(1-s)}$$

and

$$\tilde{b} = \min \left\{ -\frac{s}{R^*[\theta[(1-s)\alpha+s]-s]}, \frac{(1-\theta)R^*-\theta+\theta\sqrt{1-R^*}}{R^*[(1-\theta)^2R^*-\theta(2-3\theta)]} \right\}$$

In order to simplify the formulation, we focus on the standard value $s = 1/3$ for the share of capital in total income. We then get:

⁹See Nourry and Venditti [11] for an illustration of such a policy in the Galor-type OLG model.

Proposition 6. *Under Assumptions 1-2, let $s = 1/3$. There exist $\tilde{\alpha} \in (0, \underline{\alpha})$ and $\gamma_F > \gamma_H > 1$ such that if $\alpha \in (\tilde{\alpha}, \underline{\alpha})$, $\theta \in (s/[(1-s)\alpha + s], \tilde{\theta})$ and $b < \tilde{b}$, the NSS is locally indeterminate when $\gamma \in (\gamma_H, \gamma_F)$, undergoes a flip bifurcation when $\gamma = \gamma_F$ and undergoes a Hopf bifurcation when $\gamma = \gamma_H$. Moreover there generically exist locally indeterminate (resp. saddle-point stable) period-two cycles when $\gamma \in (\gamma_F, \gamma_F + \varepsilon)$ (resp. $\gamma \in (\gamma_F - \varepsilon, \gamma_F)$) with $\varepsilon > 0$, i.e. when the bifurcation is super (resp. sub-) critical, and locally indeterminate (resp. locally unstable) quasi-periodic cycles when $\gamma \in (\gamma_H - \epsilon, \gamma_H)$ (resp. $\gamma \in (\gamma_H, \gamma_H + \epsilon)$) with $\epsilon > 0$, i.e. when the bifurcation is super (resp. sub-) critical.*

Proof: See Appendix 6.5. □

Propositions 5 and 6 prove that the existence of expectations-driven fluctuations are more likely to occur under dynamic inefficiency and thus confirm the crucial role of the saving behavior. Moreover, in this case, any stabilization policy targeting the steady state leaves room for welfare losses associated with productive inefficiency.

5 Concluding comments

We have considered a two-sector OLG economy with two consumption goods which enter the utility function in both periods of life through a composite good. To simplify we have assumed that the share of each consumption good into the composite one is constant. We have proved that although dynamically efficient competitive equilibria are less likely to be locally indeterminate than dynamically inefficient ones, the existence of Pareto optimal expectations-driven fluctuations becomes compatible with standard sectoral technologies if the share of the pure consumption good is low enough.

6 Appendix

6.1 Proof of Proposition 1

Let $(k^*, y^*) \in (0, \bar{k}) \times (0, \bar{k})$. Solving the second equation in (13) with respect to θ and using the homogeneity of τ gives

$$\theta(k^*, y^*) = \frac{T(k^*, y^*)}{T(k^*, y^*) - T_2(k^*, y^*)(y^* - k^*)} \in (0, 1)$$

Using (10) and solving the first equation in (13) with respect to B gives

$$B(k^*, y^*, \gamma) = -\frac{T_1(k^*, y^*)}{T_2(k^*, y^*)} \left(\frac{-k^* T_2(k^*, y^*)}{\delta^\gamma [T_3(k^*, y^*) + k^* T_2(k^*, y^*)]} \right)^{\frac{1}{1-\gamma}}$$

Then (k^*, y^*) is a NSS if and only if $\theta = \theta(k^*, y^*)$ and $B = B(k^*, y^*, \gamma)$. □

6.2 Proof of Proposition 3

From (10), we derive

$$\alpha'(v) = (1 - \gamma)\alpha(v)(1 - \alpha(v))/v \quad (18)$$

Under Assumption 1, we get from the first order conditions of program (2): $T_{12} = -T_{11}b < 0$, $T_{22} = T_{11}b^2 < 0$, $T_{31} = -T_{11}a > 0$ and $T_{32} = T_{11}ab$, with $a \equiv k^0/l^0 > 0$, b as defined by (16) and $T_{11} < 0$. Consider ϵ_{rk} as given by (16) together with $T_1 k^*/T_3 = s/(1-s)$, $-T_1/T_2 = R^* = s/(1-\alpha)(1-s)$ and the fact that the homogeneity of $\tau(k, y, \ell)$ implies $a = [(1-\alpha)(1-s)(1-\theta b) - (1-\theta)b] k^*/[(1-\alpha)(1-s)]$. Total differentiation of (12) using (3), (10), (14) and (18) evaluated at the NSS gives the characteristic polynomial $\mathcal{P}_\gamma(\lambda) = \lambda^2 - \lambda \mathcal{T}_\theta(\gamma) + \mathcal{D}_\theta(\gamma)$ with

$$\begin{aligned} \mathcal{D}_\theta(\gamma) &= \frac{s \left\{ b \left[(1-s)\alpha(\gamma-1)\theta + \theta(1-s)\alpha - s(1-\theta) \right] + (1-\alpha)(1-s) \right\}}{\alpha(\gamma-1)(1-\alpha)(1-s)^2\theta b} \\ \mathcal{T}_\theta(\gamma) &= \frac{1+\alpha(\gamma-1)\epsilon_{rk} \left\{ [1-(1-\theta)R^*b]^2 + \theta^2 b^2 R^* \right\} + \theta R^* b \epsilon_{rk} \left\{ (1-\theta) \frac{b\alpha}{s} + \theta b\alpha + 1 - \alpha \right\}}{\alpha(\gamma-1)[1-(1-\theta)R^*b]\theta b \epsilon_{rk}} \end{aligned}$$

When $\theta = \theta(k^*, y^*)$ and $B = B(k^*, y^*, \gamma)$, the NSS and α remain constant as γ is made to vary, and as in Grandmont *et al.* [9], we can study the variations of $\mathcal{T}_\theta(\gamma)$ and $\mathcal{D}_\theta(\gamma)$ in the $(\mathcal{T}, \mathcal{D})$ plane. Solving \mathcal{T} and \mathcal{D} with respect to $\alpha(\gamma-1)$ yields the following linear relationship $\Delta(\mathcal{T})$:

$$\mathcal{D} = \Delta(\mathcal{T}) = \mathcal{S}_\theta \mathcal{T} + \mathcal{M} \quad (19)$$

where \mathcal{M} is a constant term and the slope \mathcal{S}_θ of $\Delta(\mathcal{T})$ is

$$\mathcal{S}_\theta = \frac{\epsilon_{rk} s R^* [1-(1-\theta)R^*b][1-s]\alpha + s[(\theta-\theta_1)(b-b_1)]}{(1-s)\{s + \theta R^* b \epsilon_{rk} \alpha [1-\theta + \theta s](b-b_2)\}} \quad (20)$$

with

$$\theta_1 = \frac{s}{(1-s)\alpha + s}, \quad b_1 = -\frac{s}{R^*[(1-s)\alpha + s](\theta-\theta_1)} \quad \text{and} \quad b_2 = -\frac{(1-\alpha)s}{\alpha(1-\theta + \theta s)} \quad (21)$$

For a given $\theta = \theta(k^*, y^*)$, as γ spans the interval $(1, +\infty)$, $\mathcal{T}_\theta(\gamma)$ and $\mathcal{D}_\theta(\gamma)$ vary linearly along the line $\Delta(\mathcal{T})$.

As $\gamma \in (1, +\infty)$, the fundamental properties of $\Delta(\mathcal{T})$ are characterized from the consideration of its extremities. The starting point of the pair $(\mathcal{T}_\theta(\gamma), \mathcal{D}_\theta(\gamma))$ is indeed obtained when $\gamma = +\infty$:

$$\begin{aligned} \lim_{\gamma \rightarrow +\infty} \mathcal{D}_\theta(\gamma) &= \mathcal{D}^\infty = \frac{s}{(1-\alpha)(1-s)} \\ \lim_{\gamma \rightarrow +\infty} \mathcal{T}_\theta(\gamma) &= \mathcal{T}_\theta^\infty = \frac{[1-R^*b(1-\theta)]^2 + \theta^2 b^2 R^*}{\theta b [1-R^*b(1-\theta)]} \end{aligned} \quad (22)$$

while the end point is obtained when γ converges to 1 from above:

$$\begin{aligned} \lim_{\gamma \rightarrow 1^+} \mathcal{D}_\theta(\gamma) &= \mathcal{D}_\theta^1 = \pm\infty \quad \Leftrightarrow \quad b(\theta - \theta_1)(b - b_1) \gtrless 0 \\ \lim_{\gamma \rightarrow 1^+} \mathcal{T}_\theta(\gamma) &= \mathcal{T}_\theta^1 = \pm\infty \quad \Leftrightarrow \quad b\mathcal{S}_\theta \lessgtr 0 \end{aligned} \quad (23)$$

$\mathcal{D}^\infty \geq 1$ for any $\theta \in (0, 1)$ if and only if $\alpha \geq \underline{\alpha}$, with $\underline{\alpha} > 0$ if and only if $s < 1/2$, while $\mathcal{T}_\theta^\infty$ depends on the value of θ . Depending on θ , the pairs $(\mathcal{T}_\theta^\infty, \mathcal{D}^\infty)$ are located on an horizontal line above (below) the line $\mathcal{D} = 1$ when $\alpha > (<) \underline{\alpha}$. Therefore $\Delta(\mathcal{T})$ is a half-line starting on the horizontal line $(\mathcal{T}_\theta^\infty, \mathcal{D}^\infty)$ and pointing upward or downward depending on $\mathcal{D}_\theta^1 = \pm\infty$.

Let us now prove Proposition 3. We conclude from (23) that if $b \in (0, b_{Max})$ then $\mathcal{D}_\theta^1 = +\infty$, $\mathcal{T}_\theta^\infty > 0$ and $\mathcal{S}_\theta > 0$. The characteristic roots are always positive. Let us finally compute

$$\mathcal{P}_\infty(1) = 1 - \mathcal{T}_\theta^\infty + \mathcal{D}^\infty = -\frac{[1-R^*b]\{1-b[R^*(1-\theta)+\theta]\}}{b\theta[1-R^*b(1-\theta)]}$$

Straightforward computations show that $\mathcal{P}_\infty(1) > 0$ if and only if $\alpha > \underline{\alpha}$, $\theta > (1-s)/[(1-s)\alpha + s]$ and $b \in (1/R^*, b_{Max})$. Moreover, when $s < 1/2$ and $\alpha < \underline{\alpha}$, if $\mathcal{D}_\theta(\gamma) = 1$ we get $\mathcal{T}_\theta(\gamma) > 2$. All this implies that for any $\alpha \in (0, 1)$, any equilibrium path is unique and monotone when $b \in (0, b_{Max})$. \square

6.3 Proof of Proposition 4

Let $b < 0$ so that $\mathcal{P}_\infty(1) > 0$. Since $\mathcal{T}_\theta^\infty < 0$, we need to compute

$$\mathcal{P}_\infty(-1) = 1 + \mathcal{T}_\theta^\infty + \mathcal{D}^\infty = \frac{[1-R^*b(1-2\theta)]\{1-b[R^*(1-\theta)-\theta]\}}{b\theta[1-R^*b(1-\theta)]}$$

Consider the bounds defined in (21) and let

$$\theta_2 = \frac{s}{1-\alpha(1-s)}, \quad b_3 = \frac{1}{R^*(1-2\theta)} \quad \text{and} \quad b_4 = \frac{1}{R^*(1-\theta)-\theta} \quad (24)$$

When $\alpha > \underline{\alpha}$, $\mathcal{P}_\infty(-1) > 0$ if and only if i) $\theta \in (1/2, \theta_2)$ and $b < b_3$, or, ii) $\theta > \theta_2$ and $b \in (b_4, b_3)$. Moreover, we derive from (23) that $\mathcal{D}_\theta^1 = -\infty$ if and only if a) $\theta < \theta_2$, or, b) $\theta > \theta_2$ and $b \in (b_1, 0)$. Finally, we get from (20) that under the conditions a) or b), $\mathcal{S}_\theta < 0$ if and only if $b > b_2(> b_1)$ and $\epsilon_{rk} > \hat{\epsilon}_{rk} = -s/[\theta R^* \alpha b(1-\theta+\theta s)(b-b_2)]$.

Using the expressions of $\mathcal{D}_\theta(\gamma)$ and $\mathcal{T}_\theta(\gamma)$ allows to show that when $\mathcal{D}_\theta(\gamma) = 1$, $\mathcal{T}_\theta(\gamma) > -2$ if and only if

$$1 + \epsilon_{rk} \left[\frac{s[(1-s)\alpha+s](\theta-\theta_1)(b-b_1)}{b(1-s)^2\theta(\underline{\alpha}-\alpha)} \mathcal{P}(b) + \frac{\theta R^* b \alpha [1-\theta+\theta s]}{s} (b-b_2) \right] < 0 \quad (25)$$

with

$$\mathcal{P}(b) = b^2 R^* [(1-\theta)^2 R^* - \theta(2-3\theta)] - 2b [(1-\theta)R^* - \theta] + 1 > 0 \quad (26)$$

for all $\theta \in (0, 1]$. Under $b < b_3$ and conditions a) or b), $\mathcal{T}_\theta(\gamma) < -2$ and $\mathcal{S}_\theta > 0$ as long as $b \leq b_2$ or $b \in (b_2, 0)$ and $\epsilon_{rk} \in (0, \hat{\epsilon}_{rk})$.

Based on all this, we now prove Proposition 4. Let $\alpha > s/(1-s)(> \underline{\alpha})$ so that $\theta_2 > 1/2 > \theta_1$.

1 - Assume that $\theta = \theta_1 + \varepsilon \leq 1/2$ with $\varepsilon > 0$ small and $b > b_2(> b_1)$. It follows that $\mathcal{P}_\infty(-1) < 0$ and the term between brackets in (25) is negative. Moreover $\mathcal{T}_\theta(\gamma) \geq -2$ if and only if

$$\epsilon_{rk} \geq -\frac{1}{\frac{s[(1-s)\alpha+s](b-b_1)}{b(1-s)^2\theta(\underline{\alpha}-\alpha)}\mathcal{P}(b)\epsilon + \frac{\theta R^*b\alpha[1-\theta+\theta s]}{s}(b-b_2)} \equiv \underline{\epsilon}_{rk}(> \hat{\epsilon}_{rk}) \quad (27)$$

Moreover, there exists $\bar{\epsilon}_{rk} > \underline{\epsilon}_{rk}$ such that $\mathcal{T}_\theta(\gamma) = 2$ when $\epsilon_{rk} = \bar{\epsilon}_{rk}$ and the half-line is given by $\bar{\Delta}$. Therefore $\mathcal{T}_\theta(\gamma) \in (-2, 2)$ as long as $\epsilon_{rk} \in (\underline{\epsilon}_{rk}, \bar{\epsilon}_{rk})$. Let us denote $\hat{\epsilon}$ the value of ϵ such that the denominator of the ratio in (27) is equal to zero. The maximal admissible value of ϵ is such that $\bar{\epsilon} = \min\{\hat{\epsilon}, 1/2 - \theta_1\}$. It follows that when $\theta \in (\theta_1, \bar{\theta})$ with $\bar{\theta} = \theta_1 + \bar{\epsilon}$ and $\epsilon_{rk} \in (\underline{\epsilon}_{rk}, \bar{\epsilon}_{rk})$, we get a half-line above Δ_1 as shown on Figure 1. This proves the first part of Proposition 4.

2 - Note from (20) that $\lim_{\epsilon_{rk} \rightarrow \hat{\epsilon}_{rk}} \mathcal{S}_\theta = \infty$. When $\epsilon_{rk} = \hat{\epsilon}_{rk}$, the half-line is thus given by $\hat{\Delta}$ and there exists $\tilde{\epsilon}_{rk} \in (\hat{\epsilon}_{rk}, \underline{\epsilon}_{rk})$ such that when $\theta \in (\theta_1, \bar{\theta})$ and $\epsilon_{rk} = \tilde{\epsilon}_{rk}$, we get a half-line such that $\tilde{\Delta}$. As a result, if $\epsilon_{rk} \in (\tilde{\epsilon}_{rk}, \underline{\epsilon}_{rk})$ the half-line is given by Δ_2 as shown on Figure 1. This proves the second part of Proposition 4.

The bifurcation values γ_H , γ_F and γ_T are respectively defined as the solutions of

$$\mathcal{D}_\theta(\gamma) = 1, \mathcal{P}_\gamma(-1) = 1 + \mathcal{T}_\theta(\gamma) + \mathcal{D}_\theta(\gamma) = 0, \mathcal{P}_\gamma(1) = 1 - \mathcal{T}_\theta(\gamma) + \mathcal{D}_\theta(\gamma) = 0$$

with the corresponding values of θ and ϵ_{rk} .

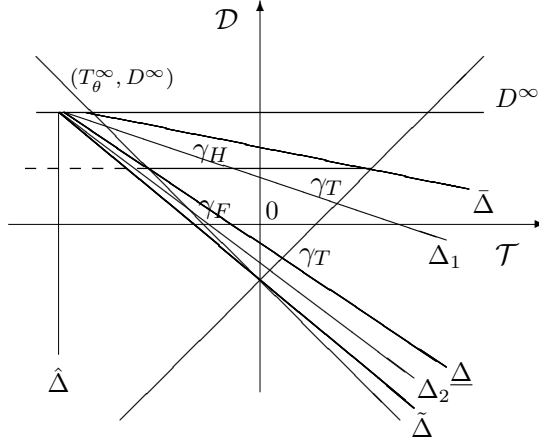


Figure 1: Hopf and flip bifurcations with $\alpha > s/(1-s)(> \underline{\alpha})$. □

6.4 Proof of Proposition 5

Consider the bounds defined in (21) and (24). $\alpha < \underline{\alpha}$ implies $\mathcal{D}^\infty < 1$. We get $\theta_2 < 1/2 < \theta_1$ and $\mathcal{P}_\infty(-1) > 0$ if and only if i) $\theta \in (\theta_2, 1/2)$ and $b < b_4$, or, ii) $\theta > 1/2$ and $b \in (b_3, b_4)$. Moreover, $\mathcal{D}_\theta^1 = +\infty$ if and only if $\theta > \theta_1$ and $b < b_1$ with $b_1 < b_3 < b_4$. This implies that when $\mathcal{P}_\infty(-1) > 0$ then

$\mathcal{D}_\theta^1 = -\infty$ and any Hopf bifurcation is ruled out. Also, when $\mathcal{D}_\theta(\gamma) = -1$, $\mathcal{T}_\theta(\gamma) < 0$ if $b < b_2$ with $b_2 > b_3$ if

$$\theta < \theta_3 = \frac{s^2 + \alpha(1-s)}{2s^2 + \alpha(1-s)^2} \in (1/2, 1)$$

Based on all this, we now prove Proposition 5. We focus here on the case in which $\mathcal{P}_\infty(1) > 0$ and $\mathcal{P}_\infty(-1) > 0$. As $b < 0$ we already know that $\mathcal{P}_\infty(1) > 0$.

i) Assume first that $\theta \in (\theta_2, 1/2)$ and $b < b_4$. This implies $\mathcal{P}_\infty(-1) > 0$ and thus $\mathcal{D}_\theta^1 = -\infty$. A flip bifurcation occurs if $\mathcal{T}_\theta(\gamma) < 0$ when $\mathcal{D}_\theta(\gamma) = -1$. This is obtained provided $b < b_2$. Part i) is then proved by assuming $b < \bar{b}$ with $\bar{b} = \min\{b_2, b_4\}$. We get indeed the half-line Δ as shown on Figure 2.

ii) Assume now that $\theta \in (1/2, \theta_3)$ and $b \in (b_3, \bar{b})$. We get as previously $\mathcal{P}_\infty(-1) > 0$, $\mathcal{D}_\theta^1 = -\infty$ and $\mathcal{T}_\theta(\gamma) < 0$ when $\mathcal{D}_\theta(\gamma) = -1$, which proves part ii).

In both cases the flip bifurcation value γ_F is defined as the solution of $\mathcal{P}_\gamma(-1) = 1 + \mathcal{T}_\theta(\gamma) + \mathcal{D}_\theta(\gamma) = 0$.

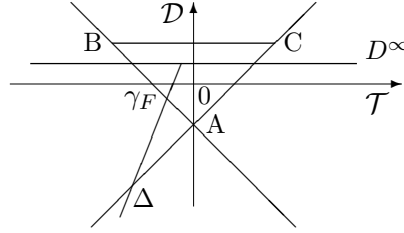


Figure 2: Flip bifurcation with $\alpha < \underline{\alpha}$. □

6.5 Proof of Proposition 6

Consider the bounds defined in (21) and (24). Let $\theta > \theta_1$ and $b < b_1 (< b_3 < b_4)$ so that $\mathcal{D}_\theta^1 = +\infty$ and $\mathcal{P}_\infty(-1) < 0$. Moreover $\mathcal{S}_\theta < 0$ as $b_1 < b_2$. Local indeterminacy may arise as shown by Δ on Figure 3 only if $\mathcal{T}_\theta^\infty \in (-2, 0)$, i.e. only if $\mathcal{P}(b) < 0$ with $\mathcal{P}(b)$ as given by (26).

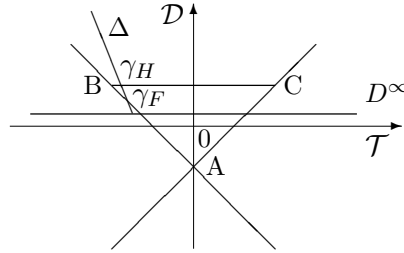


Figure 3: Flip and Hopf bifurcations with $\alpha \in (\tilde{\alpha}, \underline{\alpha})$.

Let us then consider

$$\theta_{\pm} = \frac{s+(1-\alpha)(1-s) \pm (1-s)\sqrt{(\underline{\alpha}-\alpha)(1-\alpha)}}{s+3(1-\alpha)(1-s)} \text{ and } b_+ = \frac{(1-\theta)R^* - \theta + \theta\sqrt{1-R^*}}{R^*[(1-\theta)^2 R^* - \theta(2-3\theta)]}$$

It is easy to show that $(1-\theta)^2 R^* - \theta(2-3\theta) < 0$ if and only if $\theta \in (\theta_-, \theta_+)$ with $\theta_1 > \theta_-$. Moreover, $\theta_1 < \theta_+$ if and only if $g(\alpha) = [(1-s)\alpha + s]\sqrt{(\underline{\alpha}-\alpha)(1-\alpha)} - \alpha^2(1-s) + \alpha(1+2s) - 2s > 0$, with $g(0) < 0$ and $g(\underline{\alpha}) \leq 0$ under Assumption 3. When $s = 1/3$, we get $\lim_{\alpha \rightarrow \underline{\alpha}} g'(\alpha) < 0$. It follows that there exists $\alpha_1 \in (0, \underline{\alpha})$ such that when $\alpha \in (\alpha_1, \underline{\alpha})$, $g(\alpha) > 0$. As a result we conclude that when $\alpha \in (\alpha_1, \underline{\alpha})$, $\theta \in (\theta_1, \theta_+)$ and $b < b_+$, we get $\mathcal{P}(b) < 0$. Therefore assuming $\theta \in (\theta_1, \theta_+)$ and $b < \min\{b_1, b_+\}$, we derive from (25) that there exists $\tilde{\alpha} \in [\alpha_1, \underline{\alpha})$ such that if $\alpha \in (\tilde{\alpha}, \underline{\alpha})$, $\mathcal{T}_{\theta}(\gamma) > -2$ when $\mathcal{D}_{\theta}(\gamma) = 1$. Proposition 6 is then proved under the following conditions: $\alpha \in (\tilde{\alpha}, \underline{\alpha})$, $\theta \in (s/[(1-s)\alpha + s], \tilde{\theta})$ and $b < \tilde{b}$ with $\tilde{\theta} = \theta_+$ and $\tilde{b} = \min\{b_1, b_+\}$. We get indeed the half-line Δ as shown on Figure 3. The bifurcation values γ_H and γ_F are finally defined as the solutions of

$$\mathcal{D}_{\theta}(\gamma) = 1 \text{ and } \mathcal{P}_{\gamma}(-1) = 1 + \mathcal{T}_{\theta}(\gamma) + \mathcal{D}_{\theta}(\gamma) = 0 \quad \square$$

References

- [1] Azariadis, C. (1981): “Self Fulfilling Prophecies,” *Journal of Economic Theory* 25, 380-396.
- [2] Benhabib, J. and K. Nishimura (1981): “Stability of Equilibrium in Dynamic Models of Capital Theory,” *International Economic Review* 22, 275-293.
- [3] Cass, D., and K. Shell (1983): “Do Sunspots Matter ?,” *Journal of Political Economy* 91, 193-227.
- [4] Cazzavillan, G. and P. Pintus (2007): “Dynamic Inefficiency in an Overlapping Generations Economy with Production,” *Journal of Economic Theory* 137, 754-759.
- [5] Drugeon, J.P. (2004): “On Consumptions, Inputs and Outputs Substitutabilities and the Evanescence of Optimal Cycles,” *Journal of Difference Equations and Applications* 10, 473-487 .
- [6] Drugeon, J.P. (2010): “On Sectoral Supply Functions and Heterogeneous Consumptions in a Simple Life-Cycle Economy,” mimeo CES.

- [7] Drugeon, J.P., C. Nourry and A. Venditti (2010): "On Efficiency and Local Uniqueness in Two-Sector OLG Economies," *Mathematical Social Sciences* 59, 120-144.
- [8] Galor, O. (1992): "A Two-Sector Overlapping-Generations Model: a Global Characterization of the Dynamical System," *Econometrica* 60, 1351-1386.
- [9] Grandmont, J.M., P. Pintus and R. De Vilder (1998): "Capital-Labor Substitution and Competitive Nonlinear Endogenous Business Cycles," *Journal of Economic Theory* 80, 14-59.
- [10] Kalra, S. (1996): "Cyclical Equilibria in Multi-Sector Productive Economies: the Role of Substitution and Factor Intensity," *Journal of Economic Dynamics and Control* 20, 1709-1729.
- [11] Nourry, C. and A. Venditti (2010): "Local Indeterminacy Under Dynamic Efficiency in a Two-Sector Overlapping Generations Economy," forthcoming in *Journal of Mathematical Economics*.
- [12] Reichlin, P. (1986): "Equilibrium Cycles in an Overlapping Generations Economy with Production," *Journal of Economic Theory* 40, 89-102.
- [13] Reichlin, P. (1992): "Endogenous Fluctuations in a Two-Sector Overlapping Generations Economy," in Benhabib J. (ed.), *Cycles and Chaos in Economic Equilibrium*, Princeton University Press, Princeton.
- [14] Shell, K. (1977): "Monnaie et Allocation Intertemporelle," mimeo, Séminaire d'Econométrie Roy-Malinvaut, Centre National de la Recherche Scientifique, Paris.
- [15] Venditti, A. (2005): "The Two Sector Overlapping Generations Model: a Simple Formulation," *Research in Economics* 59, 164-188.
- [16] Woodford, M. (1986): "Stationary Sunspot Equilibria. The Case of Small Fluctuations Around a Deterministic Steady State," Mimeo, University of Chicago and New York University.